Periodic Orbits and Homoclinic Loops for Surface Homeomorphisms

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Abstract

Let p be a saddle fixed point for an orientation-preserving surface diffeomorphism f, admitting a homoclinic point p'. Let V be an open 2-cell bounded by a simple loop formed by two arcs joining p to p', lying respectively in the stable and unstable curves at p. It is shown that $f|V$ has fixed point index $\rho \in \{1, 2\}$ where ρ depends only on the geometry of V near p. More generally, for every $n \geq 1$, the union of the n-periodic orbits in V is a block of fixed points for f^n whose index is ρ.

0 Introduction

Poincaré invented homoclinic orbits, conjectured their existence in the planar three body problem, and despaired of understanding their complexity. Research by Birkhoff, Cartwright & Littlewood, and Levinson revealed that near transverse homoclinic points there are robust periodic points. A pinnacle of this line of research, and the basis for much of modern dynamical theory, is Smale's "horseshoe" theorem [22]. For a diffeomorphism f of a manifold of any dimension, it states that every neighborhood of a transverse homoclinic point meets a structurally stable, hyperbolic compact invariant set K on which some iterate f^k is topologically conjugate to the shift map on the Cantor set $2^{\mathbf{Z}}$.

Similar results have been obtained under weakenings of the transversality assumption, including work by Burns & Weiss [6], Collins [8], Churchill & Rod [7], Gavrilov & Šilnikov [13, 14], Guckenheimer & Holmes [15], Mischaikow [18], Mischaikow & Mrozek [19], Newhouse [20], Rayskin [21].

Among many important consequences is the existence of hyperbolic periodic orbits in K of all periods $kn, n \geq 1$. Note, however, that k is not specified in the horseshoe theorem, and

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in most cases there is no way to estimate it (but see Mischaikow & Mrozek [19]). Collins [8] has shown that a topologically transverse homoclinic point implies the existence of periodic points of all sufficiently high minimum periods; estimating such periods, however, requires detailed knowledge of the associated homoclinic tangle.

While the horseshoe theorem guarantees infinitely many periodic orbits, it is insufficient for the existence of a second fixed point. For example, the toral diffeomorphism induced by the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ has only one fixed point, even though transverse homoclinic points are dense.

It turns out that for diffeomorphisms of the plane, even a nontransverse homoclinic point implies a second fixed point; in fact there is a block of fixed points having index $+1$. But the proof of this (Hirsch [17]), based on Brouwer's Plane Translation Theorem, gives no indication of the location of such a block.

In this paper we consider a saddle fixed point p for an orientation preserving homeomorphism f of a surface X. Let p' be a homoclinic point associated to p, i.e., a point different from p wherethe stable and unstable curves $W_s(p)$, $W_u(p)$ meet; no transversality or even crossing of these curves is assumed. Suppose $\Gamma = J_s \cup J_u$ is a homoclinic loop p, where J_s and J_u are arcs in $W_s(p)$ and $W_u(p)$ respectively, having common endpoints p, p'. Assume there is a closed 2-cell, with interior V, whose boundary is the union of two arcs in $W_s(p)$ and $W_u(p)$ respectively, having endpoints p and p' in common but otherwise disjoint; such a 2-cell always exists when X is simply connected.

Our main result, stated more precisely in Theorem 1.2, is this:

If Λ is a Jordan curve bounding an open 2-cell V, there exists $\rho \in \{1, 2\}$ such that the fixed point index of f^n in V for all $n \neq 0$, and ρ depends only on the geometry of V.

An immediate consequence is that for every $n \geq 2$, every map sufficiently close to f^n has a block of fixed points in V of index ρ .

Theorem 1.5 is a similar result for homoclinic loops that are homotopically trivial, but not necessarily Jordan curves.

The main theorem is stated in Section 1, and several applications are derived. Section 2 contains the proof of the main theorem.

1 The main result and applications

$\mathbf{Z}, \mathbf{N}$ and $\mathbf{N}_+$ denote the integers, natural numbers and positive natural numbers.

All maps are assumed continuous; $\approx$ denotes homeomorphism.

For any map g, the maps $g^n, n \geq 1$ are defined recursively by $g^1 = f$ and $g^{n+1}(x) = g(g^n(x))$ provided $g^n(x)$ is in the domain of g.

X always denotes a connected, oriented surface with metric d, and $f\colon X \to S$ is an orientation preserving injective map. We call f a diffeomorphism when f and f^{-1} are C^1 (continuously differentiable).

The *orbit* of x is the set $\gamma(x) = \{f^i(x) : i \in \mathbf{Z}\}$. The fixed point set of f is denoted by $\mathsf{Fix}\,(f)$. We call $q \in \mathsf{Fix}\,(()f)$ *smooth* if it belongs to a coordinate chart in which f is

represented by a C^1 map; such a chart is *smooth* for p. If f is C^1, of course all fixed points are smooth. But in many constructions some fixed points of a nonsmooth maps are smooth, as when a diffeomorphism of the plane is extended to the 2-sphere.

Let $q \in \mathsf{Fix}\,(f)$ be smooth. We call q *simple* if 1 is not an eigenvalue of the linear operator df_q, *hyperbolic* if no eigenvalue lies on the unit circle $S^1 \subset \mathbf{C}$, a *sink* if eigenvalues are inside S^1, a *source* if they are outside, and *elliptic* if the eigenvalues are on S^1 but different from 1.

A fixed point p is a *saddle* if it is not in the boundary of S, and there is a chart at p in which f is locally represented as a linear map $\left[\begin{smallmatrix}\mu & 0\\ 0 & \lambda\end{smallmatrix}\right]$, and either $\mu > 1 > \lambda > 0$, making p a *direct* saddle, or $\mu < -1 < \lambda < 0$, defining a *twisted saddle.* Such a chart is *diagonalizing.* By the Hartman-Grobman linearization theorem (Hartman [16]), for p to be a saddle it is sufficient for there to be a smooth chart at p in which df_p has eigenvalues μ, λ as above.

An *n-periodic* point for f means a fixed point z for f^n, $n \geq 1$. When n is the minimum period, $\gamma(z)$ is an *n-orbit.* An n-periodic point is simple, hyperbolic, and so forth, when it has the corresponding property as a fixed point for f^n.

The *stable curve* $W_s = W_s(p)$ of a saddle fixed point p is the connected component of p in the set of x for which there is a convergent sequence $x_k \to p$ in X with $x_0 = x$ and $f(x_k) = x_{k+1}$. The *unstable curve* W_u at p is defined as the stable curve for f^{-1}. Note that W_s and W_u are mapped homeomorphically onto themselves by f. Owing to the linearization assumption, there are bijective maps $\zeta_u, \zeta_s : \mathbf{R} \to W_s$ taking 0 to p, called *parametrizations* of W_u, W_s respectively. The images of $[0, \infty)$ and $(-\infty, 0]$ are the four *branches* at p.

A *homoclinic point* for p is any point $p' \in W_s \cap W_u \setminus \{p\}$, in which case the *homoclinic loop* Λ defined by p' is the closed path formed by the two arcs $J_s \subset W_s$, $J_u \subset W_u$ having common endpoints p and p'. There corresponds an element $[\Lambda]$ of the fundamental group of X at p, determined by first traversing Λ from p to p' in J_u and then from p' to p in J_s. If $[\Lambda]$ is the unit element, Λ is a *inessential homoclinic loop.* Λ is *simple* if $J_u \cap J_s = \{p, p'\}$, in which case Λ is homeomorphic to the unit circle. Every homoclinic loop contains a simple homoclinic loop.

Suppose Λ is a simple homoclinic loop in X bounding a closed 2-cell $D \subset X$. The corresponding open 2-cell $V = D \setminus \partial D$ is a *homoclinic cell.* We call V is a *positive* cell provided some diagonalizing chart takes p to the origin $0 \in \mathbf{R}^2$ and a neighborhood of p in D onto a neighborhood of 0 in the first quadrant. In the contrary case D is a *negative* cell: there is a diagonalizing chart taking a neighborhood of p in D onto a neighborhood of the origin in the complement of the open first quadrant (see Figure 1). Thus when seen through a diagonalizing chart, a positive cell appears convex near p, while a negative region appears concave.

Let $U \subset X$ be an open set such that $U \cap \mathsf{Fix}\,(f)$ is compact. The *fixed point index of f in U* is denoted by $I(f, U) \in \mathbf{Z}$; if it is nonzero, there exists a fixed point in U (Dold [9]). When U is a coordinate chart identified with an open set in $\mathbf{R}^2$, we can calculate $I(f, U)$ as follows. Let $M \subset U$ be a compact surface with boundary whose interior contains

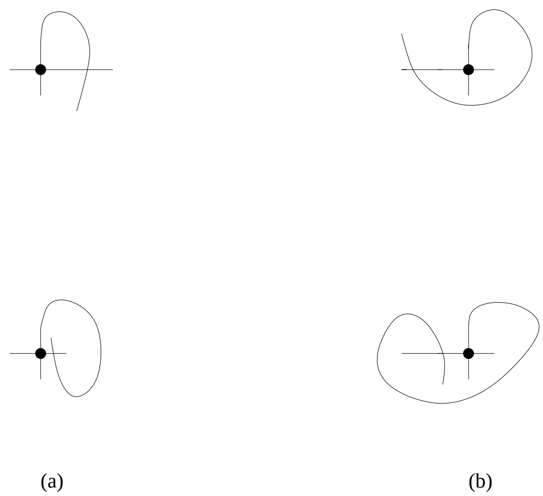

Figure 1: Homoclinic cells: (a) positive (b) negative

$\mathsf{Fix}\,(f) \cap U$. Then $I(f, U)$ is the degree of the map

$$\partial M \to S^1, \quad x \mapsto \frac{x - f(x)}{||x - f(x)||},$$

where ∂M and S^1 inherit their orientations from $\mathbf{R}^2$. If ∂M is replaced by any oriented Jordan curve Γ on which f has no fixed points, the same formula defines the *index of f along* Γ.

Let $B \subset X$ be a *block* of fixed points, i.e., B is compact and relatively open in $\mathsf{Fix}\,(f)$. There exists an open neighborhood $U_0 \subset X$ such that $B = \mathsf{Fix}\,(f) \cap U_0$. The number

$$\mathsf{Ind}(f, B) = \mathsf{Ind}(f|U_0) \in \mathbf{Z},$$

called the *index of f at B*, is independent of the choice of U_0. When p is an isolated fixed point we set $\mathsf{Ind}\, f, \{p\}) = I(f, p)$, called the *index of f at p.* A direct saddle has index -1. Twisted saddles, sources, sinks and elliptic fixed points have index $+1$.

The following assumptions are in force throughout the rest of this article:

Hypothesis 1.1

- *$f\colon X \approx X$ is an orientation-preserving homeomorphism of a surface X.*
- *$p \in X \setminus \partial X$ is a direct saddle fixed point for f.*
- *$V \subset X$ is an open 2-cell bounded by a simple homoclinic loop Λ at p.*

To V we assign the number

$$\rho = \rho(V) = \left\{ \begin{array}{ll} 1 & \text{if } V \text{ is a positive region} \\ 2 & \text{if } V \text{ is a negative region.} \end{array} \right\}$$

For each $n \in \mathbf{N}_+$ we define an open set $V_n \subset V$,

$$V_n = V_n(f) = \{x \in V : f^i(x) \in V,\ i = 1, \ldots, n-1\}$$

Thus $\mathsf{Fix}\,(f^n) \cap V_n$ is the union of the n-periodic orbits in V.

The following is our fundamental result:

Theorem 1.2 $\mathsf{Fix}\,(f^n) \cap V_n$ *is a block of fixed points for* f^n *of index* $\rho(V)$*, for all* $n \geq 1$.

Before giving the proof of Theorem 1.2 in Section 2, we present several consequences. Hypothesis 1.1 is always assumed.

Homeomorphisms of the sphere

Assume $g\colon\ S^2 \approx S^2$ is an orientation preserving homeomorphism having a simple homoclinic loop Λ at a direct saddle.

Theorem 1.3 *The fixed point index of* g *in one of the two complementary components of* Λ *is* 1*, and the index in the other is* 2.

Proof This follows from Theorem 1.2, because one complementary component of Λ has positive type and the other has negative type. ∎

The persistence of blocks having nonzero index implies:

Corollary 1.4 *Every map* $S^2 \to S^2$ *sufficiently close to* g *has at least 3 fixed points.*

A homoclinic loop constrains fixed point indices. Suppose for example that there are exactly 3 fixed points: a direct saddle and two other fixed points with respective indices 5 and -2. Then the saddle does not admit a homoclinic point.

Inessential homoclinic loops and Nielsen classes

Fixed points a, b are in the same *Nielsen class* provided they are endpoints of a path that is homotopic to its composition with f, keeping endpoints fixed. Equivalently, f is covered by a map in a universal covering space having fixed points over a and b.

When X is compact, every Nielsen class is a block of fixed points, and the *Nielsen number* of f is the number of Nielsen classes having nonzero index. This number, a homotopy invariant of f, is a lower bound for the number of fixed points for any map homotopic to f.

Theorem 1.5 *Assume p belongs to an inessential homoclinic loop. Then its Nielsen class contains a block of positive index, and such a block must contain a fixed point $q \neq p$. When the Nielsen class of p is finite, q can be chosen with positive index.*

Thus in the presence of a inessential homoclinic loop, the number of fixed points exceeds the Nielsen number. Theorem 1.10 is a similar result for Lefschetz numbers.

Corollary 1.6 *If a direct saddle p is the only member of its Nielsen class, then p does not belong to an inessential homoclinic loop.*

In proving Theorem 1.5 we assume X is not simply connected, otherwise using Theorem 1.2. As X is orientable, there is a universal covering space $\pi : \mathbf{R}^2 \to X$. Choose $\tilde{p} \in \pi^{-1}(p)$, and let $\tilde{f} \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the unique lift of f with a fixed point at $\tilde{p}$. Then $\tilde{p}$ is a direct saddle for $\tilde{f}$.

Let Γ be a null homotopic homoclinic loop at p. There is a unique homoclinic loop $\tilde{\Gamma} \subset \mathbf{R}^2$ for $\tilde{f}$ that contains $\tilde{p}$ and projects onto Γ under π. Let $\tilde{\Lambda} \subset \tilde{\Gamma}$ be a simple homoclinic loop at $\tilde{p}$. There is a unique an open 2-cell $V \subset \mathbf{R}^2$ bounded by Λ, and $\overline{V}$ is a closed 2-cell. Applying Theorem 1.2, we choose a block $L \subset \mathsf{Fix}\,(\tilde{f}) \cap V$ such that

$$\mathsf{Ind}(\tilde{f}, L) = \sigma \in \{1, 2\}$$

Notice that $\pi(L)$ lies in the Nielsen class of p.

Every fixed point $\tilde{z} \in \pi^{-1}(p)$ has index -1, since $\tilde{f}$ in a neighborhood of $\tilde{z}$ is conjugate to f in a neighborhood of $\pi(z)$. Because $\overline{V}$ is compact, $\pi^{-1}(p) \cap V$ is finite. Therefore $L \setminus \pi^{-1}(p)$ is nonempty, for otherwise L would be a nonempty finite subset of $\pi^{-1}(p)$ and thus have negative index.

It follows that $\pi(L \setminus \pi^{-1}(p))$ is a nonempty subset of $\mathsf{Fix}\,(f)$ disjoint from p, contained in the Nielsen class of p. Suppose this class is finite. Then $L \setminus \pi^{-1}(p))$ is finite. Let $L \cap \pi^{-1}(p)$ have cardinality ν, $1 \leq \nu < \infty$. Then

$$\begin{aligned}
\mathsf{Ind}(\tilde{f}, L \setminus \pi^{-1}(p))) &= \mathsf{Ind}(\tilde{f}, L) - \mathsf{Ind}(\tilde{f}, L \cap (\pi^{-1}(p))) \\
&= \mathsf{Ind}(\tilde{f}, L) - \nu \; \mathsf{Ind}(f, p)) \\
&= \sigma + \nu \quad \geq \quad 2
\end{aligned}$$

Therefore exists $\tilde{q} \in L \setminus \pi^{-1}(p)$ with $0 < \mathsf{Ind}(\tilde{f}, \tilde{q}) = \mathsf{Ind}(f, \pi(\tilde{q}))$, and $\pi(\tilde{q})$ is in the Nielsen class of p. This completes the proof of Theorem 1.5.

Periodic orbits in a homoclinic cell

The following theorem can be used to demonstrate the existence of infinitely many periodic orbits in situtations where the horseshoe theorem may not apply:

Theorem 1.7 *Let $r \in \mathbf{N}$ be such that every 2^k-orbit, $0 \leq k \leq r$ in the homoclinic cell V is hyperbolic. Then either:*

(a) *V contains an attracting or repelling 2^k-orbit for some $k \in \{0, \dots, r\}$,*

or else

(b) *V contains a twisted saddle orbit of cardinality 2^k for every $k = 0, \dots, r$.*

Proof Suppose (a) does not hold. Fix $n = 2^k$, $0 \le k \le r$ and let $B \subset \mathsf{Fix}\,(f|V_n)$ be a block having index $\rho \in \{1, 2\}$ (Theorem 1.2). Then some $q \in B$ has index 1 for f^n. Since (a) is ruled out, q is not a source or sink for f^n. The only other possibility for a hyperbolic, index 1 fixed point for f^n is a twisted saddle. This implies n is the minimal period for q. Thus (b) holds. ∎

Corollary 1.8 *Assume every periodic orbit in V whose cardinality is a power of 2 is a saddle. Then V contains a twisted saddle orbit of cardinality 2^k for every $k \in \mathbf{N}$.*

Corollary 1.9 *If f is C^1 and $0 < \mathrm{Det}\, df_x < 1$ in a dense subset of V and all periodic points in V are hyperbolic, then V contains either a periodic attractor, or an orbit of cardinality 2^k for every $k \in \mathbf{N}$*

It is interesting to compare these results to a theorem of Franks [10]. Specialized to an orientation-preserving diffeomorphism of the 2-sphere, it states:

> *If all periodic points are hyperbolic, and at most one orbit whose cardinality is a power of 2 is repelling or attracting, then there are infinitely many periodic orbits.*

Corollary 1.8 makes no assumptions on orbits outside the homoclinic cell V, but does not allow any attractors or repellors of cardinality 2^k in V. It gives sharper information than the conclusion Franks' theorem on the periods and locations of periodic orbits.

It is not trivial to construct diffeomorphisms of the disk or sphere, all of whose periodic orbits are saddles; but examples are known that even have the Kupka-Smale property, i.e., stable and unstable curves of periodic points have only transverse intersections (Bowen & Franks [3], Franks & Young [11]). Gambaudo *et. al* [12] construct real analytic Kupka-Smale examples on the disk.

Lefschetz numbers

Let $\#Q$ the cardinality of a set Q.

Suppose X is a compact surface and $h : X \to X$ is continuous. The *Lefschetz number* $\mathsf{Lef}(h)$ is the alternating sum of the traces of the induced endomorphisms of the singular homology groups $H_i(X)$, $i = 0, 1, 2$; it equals $\mathsf{Ind}(h, X)$. Lefschetz proved that when the fixed point set is finite, $\mathsf{Lef}(h)$ is the sum of the fixed point indices. When every fixed point has index $+1$, -1 or 0 this gives the useful estimate

$$\#\,\mathsf{Fix}\,(h) \ge |\mathsf{Lef}(h)|$$

The following results show that when fixed points are simple, homoclinic cells entail the existence of more fixed points than are counted by the Lefschetz number.

Theorem 1.10 *Assume X is a compact surface, $\mathsf{Fix}\,(f)$ is finite, and every fixed point has index $+1$, -1 or 0. If f admits a homoclinic cell, then*

$$\#\,\mathsf{Fix}\,(f) \;\geq\; |\mathsf{Lef}(f)\,+1-\rho|+1+\rho \;\geq\; |\mathsf{Lef}(f)|+2$$

Proof For any open set $A \subset X$, summing indices over fixed points $z \in A$ gives:

$$\begin{aligned}|\,\mathsf{Ind}(f,A)| = \left|\sum_z \mathsf{Ind}(f,z)\right| &\leq \sum_z |\,\mathsf{Ind}(f,z)| \\ &\leq \#\,(\mathsf{Fix}\,(f)\cap A)\end{aligned}$$

Applying this to a homoclinic cell V, from Theorem 1.2 we get

$$\begin{aligned}\#\,(\mathsf{Fix}\,(f)\cap \overline{V}) = 1+\#\,(\mathsf{Fix}\,(f)\cap V) &\geq 1+|\,\mathsf{Ind}(f,V)| \\ &= 1+\rho\end{aligned}$$

because $\mathsf{Ind}(f,V)=\rho$. Also

$$\begin{aligned}\#\,(\mathsf{Fix}\,(f)\cap (X\setminus\overline{V})) &\geq |\,\mathsf{Ind}(f,X\setminus\overline{V})| \\ &= |\,\mathsf{Ind}(f,X)-(\mathsf{Ind}(f,p)+\mathsf{Ind}(f,V))| \\ &= |\mathsf{Lef}(f)+1-\rho|\end{aligned}$$

because $\mathsf{Ind}(f,p)=-1$. Therefore

$$\begin{aligned}\#\,\mathsf{Fix}\,(f) &= \#\,(\mathsf{Fix}\,(f)\cap\overline{V}) \;+\; \#\,(\mathsf{Fix}\,(f)\cap(X\setminus\overline{V})) \\ &\geq (1+\rho) \;+\; |\mathsf{Lef}(f)+1-\rho| \\ &\geq |\mathsf{Lef}(f)|+2\end{aligned}$$

∎

Corollary 1.11 *Assume X is a compact surface, $\mathsf{Fix}\,(f)$ is finite, and every fixed point has index $+1$, -1 or 0. If $\#\,\mathsf{Fix}\,(f) \leq |\mathsf{Lef}(f)|+1$, there are no homoclinic cells.*

2 Fixed point indices and retractions

This section contains the proofs of Theorems 1.2 and 1.5. Hypothesis 1.1 continues to hold. Let $D \subset X$ denote the closure of the homoclinic cell V. Then D is a compact 2-cell whose boundary is the simple homoclinic loop Λ.

A *retraction* of a space Y onto a subset $Y_0 \subset Y$ is a map $Y \to Y_0$ fixing every point in Y_0.

Lemma 2.1 *Assume we are given $n \in \mathbf{N}_+$ and a map $g\colon D \to D$ with the following properties:*

(i) *g coincides with f^n on a neighborhood of p in D*

(ii) $\mathsf{Fix}\,(g) = K \cup \{p\}$ *where $K \subset V$ is compact.*

Then $\mathsf{Ind}(g, K) = \rho(V)$

Proof Fix a coordinate chart in which p is the origin and f^n is represented by a linear map

$$T(x, y) = (\lambda x, \mu y), \quad 0 < \lambda < 1 < \mu$$

We identify points near p with their images in $\mathbf{R}^2$ under this chart.

Consider the case that V is a positive homoclinic cell ($\rho = 1$). Then there is a compact disk neighborhood $N \subset \mathbf{R}^2$ centered at the origin, meeting D only in one of the four closed quadrants; to fix ideas, we assume it is the first quadrant Q_I. We take N so small that g coincides with T in $N \cap D$, $N \cap K = \varnothing$, and $N \cup D$ is a 2-cell.

Choose a retraction $s : N \to N \cap Q_I$. We compute the fixed point index $\mathsf{Ind}(T \circ s, N)$. Let $\epsilon > 0$ be so small that the disk D_ϵ of radius ϵ lies in N. Let S^1_ϵ denote the circle bounding D_ϵ. Since $T \circ s$ has the unique fixed point 0, the index equals the degree of the map

$$u : S^1_\epsilon \to S^1, \; z \mapsto \frac{z - T \circ s(z)}{||z - T \circ s(z)||}$$

The retraction s sends any point $z \in N \setminus Q_I$ to the unique point $s(z) \in \partial Q_I$ such that z and $s(z)$ are the endpoints of a line segment having slope 1; and s is the identity on $N \cap Q_I$. A simple computation shows that u takes no values in the first quadrant of the unit circle, and thus has degree zero. Thus $\mathsf{Ind}(T \circ s, N) = 0$.

Now consider the map $h : N \cup D \to D \subset N \cup D$ defined to be $T \circ s$ in N and g in D; this definition is consistent because s is a retraction and g coincides with T in $N \cap D$. Clearly

$$\mathsf{Fix}\,(h) = \{p\} \cup K \subset \mathrm{Int}(N \cup D)$$

Therefore

$$\mathsf{Lef}(h) = \mathsf{Ind}(h, \mathrm{Int}(N \cup D)) = \mathsf{Ind}(h, p) + \mathsf{Ind}(h, K)$$

Note that $\mathsf{Lef}(h) = 1$ because $N \cup D$ is a compact 2-cell, and $\mathsf{Ind}(h, p) = \mathsf{Ind}(T \circ s, N) = 0$. Hence

$$1 = \mathsf{Ind}(h, K) = \mathsf{Ind}(g, K)$$

as required.

When V is a negative homoclinic cell, we can assume $N \cap D$ *excludes* the interior of the first quadrant. The retraction $s : N \to N \setminus \mathrm{Int}\, Q_I$ is defined by sending $z \in N \cap Q_I$ to the unique point of ∂Q_I such that z and $r(z)$ are the endpoints of line segment having slope 1; and r is the identity on $N \setminus Q_I$. The degree of $T \circ r$ in this case is -1. Define h as above. An argument similar to the preceding shows that

$$\begin{aligned} 1 &= \mathsf{Ind}(h, \mathrm{Int}(N \cup D)) = \mathsf{Ind}(h, \{p\}) + \mathsf{Ind}(h, \mathrm{Int}(N \cup D)) \\ &= -1 + \mathsf{Ind}(g, \mathrm{Int}(N \cup D)) \end{aligned}$$

∎

Let $J_u \subset W_u(p)$ and $J_s \subset W_s(p)$ denote the two compact arcs whose union is Λ; these arcs meet at their common endpoints, which are p and the homoclinic point $p' \neq p$, but nowhere else.

Our next goal is the following result:

Proposition 2.2 *There is a retraction*

$$r : f(D) \cup D \to D$$

such that

$$r(f(D) \setminus D) \subset J_s \tag{1}$$

Proof We first prove

$$J_u \cap \mathrm{clos}(f(D) \setminus D) = \{p, p'\} \tag{2}$$

or equivalently,

$$J_u \cap \mathrm{clos}(f(V) \setminus D) = \{p, p'\}$$

Suppose (2) is false, so that there exists

$$b \in J_u \setminus \{p, p'\} \cap \mathrm{clos}(f(V) \setminus D)$$

Then $b = \lim_{i \to \infty} f(a_i)$ for some sequence $a_i \in V \setminus f^{-1}D$, and $b = f(a)$ by continuity. I claim f maps a relatively open neighborhood $N_a \subset D$ of a onto a relatively open neighborhood $f(N_a)) \subset f(D)$ of $f(a)$. This is because f maps the interior of D onto the interior of $f(D)$. The assumption that p is a direct saddle implies f preserves orientation, and $f^{-1}|J_u$ preserves orientation in J_u. From this it follows that N_a and $f(N_a)$ abut J_u from the same side. Consequently $f(N_a)$ contains a relatively open neighborhood $N_b \subset D$ of b. For sufficiently large i we have $a_i \in f^{-1}N_b$ and thus $a_i \in f^{-1}D$. This contradiction completes the proof of (2).

From Equation (2) we see that

$$\mathrm{clos}(f(D) \setminus D) \cap \ D \subset J_s \tag{3}$$

Note also

$$f(D) \cup D = \mathrm{clos}(f(D) \setminus D) \cup D,$$
$$\mathrm{clos}(f(D) \setminus D) \cap D = \mathrm{clos}(f(D) \setminus D) \cap \partial D \subset J_s$$

By Tietze's extension theorem there is a retraction

$$r_0 : \mathrm{clos}(f(D) \setminus D) \cup J_s \to J_s$$

and r_0 agrees with the identity map of D on the intersection of their domains, which by (3) is J_s. Thus r_0 and the identity map of D fit together to give the desired retraction r. ∎

From now on $r : f(D) \cup D \to D$ denotes a retraction as in Proposition 2.2.

Lemma 2.3 *Let $n \in \mathbf{N}$. For every $q \in \mathsf{Fix}\,(f^n) \cap V_n$ there is a neighborhood $U \subset V_n$ of q such that $f^n|U = (r \circ f)^n|U$.*

Proof The definition of V_n implies $f^j(q) \in V_n \subset V$ for all $j \in \mathbf{N}$. Therefore q has a neighborhood U such that $f^i(U) \subset V_n$ for $i = 0, \dots, n$. Assume inductively that $0 \leq i < n$ and $f^i|U = (r \circ f)^i|U$; the case $i = 0$ is trivial. For $x \in U$ we have $(r \circ f)^i(x) = f^i(x)$, and both $f^i(x)$ and $f^{i+1}(x)$ are in V because $x \in V_n$. Hence

$$(r \circ f)^{i+1}(x) = (r \circ f)(f^i(x)) = r(f^{i+1}(x)) = f^{i+1}(x)$$

because r and f coincide on V. This completes the induction. ∎

Lemma 2.4 $\mathsf{Fix}\,((r \circ f|D)^n) = \{p\} \cup (\mathsf{Fix}\,(f^n) \cap V_n)$ *for all $n \geq 1$.*

Proof Let $x \in D \setminus \{p\}$ be n-periodic for $r \circ f$. We first show $x \notin J_s$. We know that J_s is invariant under f, and $r|J_s$ is the identity because $J_s \subset D$. Thus $r \circ f|J_s$ coincides with $f|J_s$, whose only periodic point is p. The foregoing implies no point on the orbit x under $r \circ f$ lies in J_s. Therefore no point y in this orbit maps outside D under f, for otherwise $(r \circ f)(y) \in J_s$ by Equation (1). This proves $\gamma(x) \subset D$, and an induction that $(r \circ f)^k x = f^k x$ for all k. Since $J_u \setminus p$ is contains no periodic points for f, the conclusion follows. ∎

Proof of Theorem 1.2

The set $B = \mathsf{Fix}\,(f^n) \cap V_n$ is open in $\mathsf{Fix}\,(f^n)$ because V_n is open. We prove B compact by showing it is closed in D. Since $\overline{B} \cap \partial D \subset \{p\}$, it suffices to prove that p is not a limit point of B. Clearly $p \notin B$, and p, being a saddle, has a neighborhood in which the only point of period n is p. Therefore B is a block.

To prove $\mathsf{Ind}(f^n, B) = \rho$, let $r : f(D) \cup D \to D$ be a retraction as in Proposition 2.2. Lemmas 2.4 and 2.3 show that $\mathsf{Ind}(f^n, B) = \mathsf{Ind}((r \circ f|V)^n, B)$. Now apply Lemma 2.1 to $g := (r \circ f|D)^n$ to conclude that $\mathsf{Ind}((r \circ f|V)^n, B) = \rho$. ∎

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